# Access to Science, Engineering and Agriculture: Mathematics 1 MATH00030 Semester 1 2017-2018 Exam Solutions

All the exam questions are unseen.

**1.** (a) (i) $\frac{3}{5} - \frac{4}{3} = \frac{(3)(3) - (5)(4)}{(5)(3)} = \frac{-11}{15} = -\frac{11}{15}$.

(ii) $\frac{3}{5} \times \left(-\frac{7}{11}\right) = \frac{(3)(-7)}{(5)(11)} = \frac{-21}{55} = -\frac{21}{55}$.

(iii) $\frac{2}{7} \div \frac{9}{13} = \frac{2}{7} \times \frac{13}{9} = \frac{(2)(13)}{(7)(9)} = \frac{26}{63}$.

(iv) $-2^2 = -\left(2^2\right) = -4$.

(v) $\left(\frac{27}{64}\right)^{-\frac{2}{3}} = \frac{1}{\left(\frac{27}{64}\right)^{\frac{2}{3}}} = \frac{1}{\left(\left(\frac{27}{64}\right)^{\frac{1}{3}}\right)^2} = \frac{1}{\left(\frac{3}{4}\right)^2} = \frac{1}{9/16} = \frac{16}{9}$.

(vi) $(3 \times (-4) - 5 \div (-6))^2 = \left(-12 + \frac{5}{6}\right)^2 = \left(-\frac{67}{6}\right)^2 = \frac{4489}{36}$.

(vii) Since $5^3 = 125$, it follows that $\log_5 125 = 3$.

(viii) Since $8^{-\frac{1}{3}} = \frac{1}{2}$, it follows that $\log_8 \frac{1}{2} = -\frac{1}{3}$. **[8]**

(b) (i) $x^{-7} \times x^4 = x^{-7+4} = x^{-3}$.

(ii) $x^{-\frac{1}{3}} \div x^{\frac{2}{5}} = x^{-\frac{1}{3}-\frac{2}{5}} = x^{-\frac{11}{15}}$.

(iii) $\left(x^{-3}\right)^2 = x^{-3(2)} = x^{-6}$. **[3]**

(c)

$$\begin{aligned}
\log_a\left(\left(\frac{x^2}{y^3}\right)^{-3}\right) &= -3\log_a\left(\frac{x^2}{y^3}\right) \\
&= -3\left[\log_a\left(x^2\right) - \log_a\left(y^3\right)\right] \\
&= -3\left[2\log_a x - 3\log_a y\right] \\
&= -6\log_a x + 9\log_a y.
\end{aligned}$$

**[2]**

(d) (i) $9.99999 = 10.000$ to three decimal places.

(ii) $0.0004499 = 0.0004$ to one significant figure.

(iii) $0.000456274 = 4.56274 \times 10^{-4}$ in scientific notation.

(iv) $24901624 = 2.5 \times 10^7$ in scientific notation to two significant figures. **[4]**

(e)

$$
\begin{aligned}
(-x^2 + 2x - 2) - (x^2 + 4x - 4) &= (-x^2 - x^2) + (2x - 4x) + (-2 + 4) \\
&= -2x^2 - 2x + 2.
\end{aligned}
$$

**[1]**

(f)

$$
\begin{aligned}
(2x^4 - 4x)(-x^2 + 2) &= (2x^4)(-x^2 + 2) + (-4x)(-x^2 + 2) \\
&= (2x^4)(-x^2) + (2x^4)(2) + (-4x)(-x^2) + (-4x)(2) \\
&= -2x^{4+2} + 4x^4 + 4x^{1+2} - 8x \\
&= -2x^6 + 4x^4 + 4x^3 - 8x.
\end{aligned}
$$

**[2]**

(g)

$$
\begin{array}{r|l}
 & x - 1 \\
\hline
x - 2 & x^2 - 3x - 1 \\
 & -x^2 + 2x \\
\hline
 & -x - 1 \\
 & x - 2 \\
\hline
 & -3
\end{array}
$$

This tells us that $\dfrac{x^2 - 3x - 1}{x - 2} = (x - 1) + \dfrac{-3}{x - 2}$.
So the quotient is $x - 1$ and the remainder is $-3$. **[4]**

(h) $\sum_{i=-2}^{2} -ix^i = -(-2)x^{-2} - (-1)x^{-1} - (0)x^0 - 1x^1 - 2x^2 = 2x^{-2} + x^{-1} - x - 2x^2$. **[2]**

(i) $\binom{10}{8} = \binom{10}{2} = \dfrac{10 \times 9}{2 \times 1} = 45.$ **[2]**

(j)

$$
\begin{aligned}
(2x + 3y^2)^3 &= (2x)^3 + \binom{3}{1}(2x)^2(3y^2) + \binom{3}{2}(2x)(3y^2)^2 + (3y^2)^3 \\
&= 8x^3 + 36x^2y^2 + 54xy^4 + 27y^6.
\end{aligned}
$$

**[4]**

**2.** (a) Here our line is parallel to a line that has slope 2, so our line also has slope $m = 2$.
Hence the equation of the line is $y = 2x + c$, where we still have to find $c$.
On substituting $x = -1$ and $y = -3$ into $y = 2x + c$, we obtain $c = -3 - 2(-1) = -1$.
Hence the equation of the line is $y = 2x - 1$. **[2]**

(b) If we add two times the second equation to three times the first we obtain

$$
\begin{array}{rrrrr}
 & 6x & + & -15y & = & 54 \\
+ & -6x & + & -8y & = & -8 \\
\hline
 & & & -23y & = & 46
\end{array}
$$

Hence $y = -2$ and on substituting this into the first equation we get
$2x - 5(-2) = 18$, so that $2x = 18 - 10 = 8$ and hence $x = 4$.
Thus the solution is $x = 4$ and $y = -2$. **[3]**

(c) Using $(x_1, y_1) = (-2, -3)$ and $(x_2, y_2) = (-4, 2)$, the formula tells us that the length of the line segment is

$$\sqrt{(-4-(-2))^2 + (2-(-3))^2} = \sqrt{(-2)^2 + 5^2} = \sqrt{4+25} = \sqrt{29}.$$

**[1]**

**3.** (a)

$$\begin{aligned} 3x^2 - 4x + 1 &= 3\left\{x^2 - \frac{4}{3}x + \frac{1}{3}\right\} \\ &= 3\left\{\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} + \frac{1}{3}\right\} \\ &3\left\{\left(x - \frac{2}{3}\right)^2 - \frac{1}{9}\right\} \\ &= 3\left(x - \frac{2}{3}\right)^2 - \frac{1}{3}. \end{aligned}$$

**[3]**

(b)

$$\begin{aligned} 3x^2 - 4x + 1 = 0 &\Rightarrow 3\left(x - \frac{2}{3}\right)^2 - \frac{1}{3} = 0 \\ &\Rightarrow 3\left(x - \frac{2}{3}\right)^2 = \frac{1}{3} \\ &\Rightarrow \left(x - \frac{2}{3}\right)^2 = \frac{1}{9} \\ &\Rightarrow x - \frac{2}{3} = \pm\frac{1}{3} \\ &\Rightarrow x = \frac{1}{3} \text{ or } x = 1. \end{aligned}$$

**[2]**

(c) From Part (b) we know that the graph cuts the $x$-axis when $x = \frac{1}{3}$ and when $x = 1$. Next, when $x = 0$, $y = 1$, so the graph cuts the $y$-axis when $y = 1$.
We also know the graph is U-shaped since $a > 0$.
Finally, the turning point is given by

$$\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right) = \left(-\frac{-4}{2(3)}, -\frac{(-4)^2 - 4(3)(1)}{4(3)}\right) = \left(\frac{2}{3}, -\frac{1}{3}\right).$$

We now have all the information we need and I have sketched the graph in Figure 1.

**[4]**

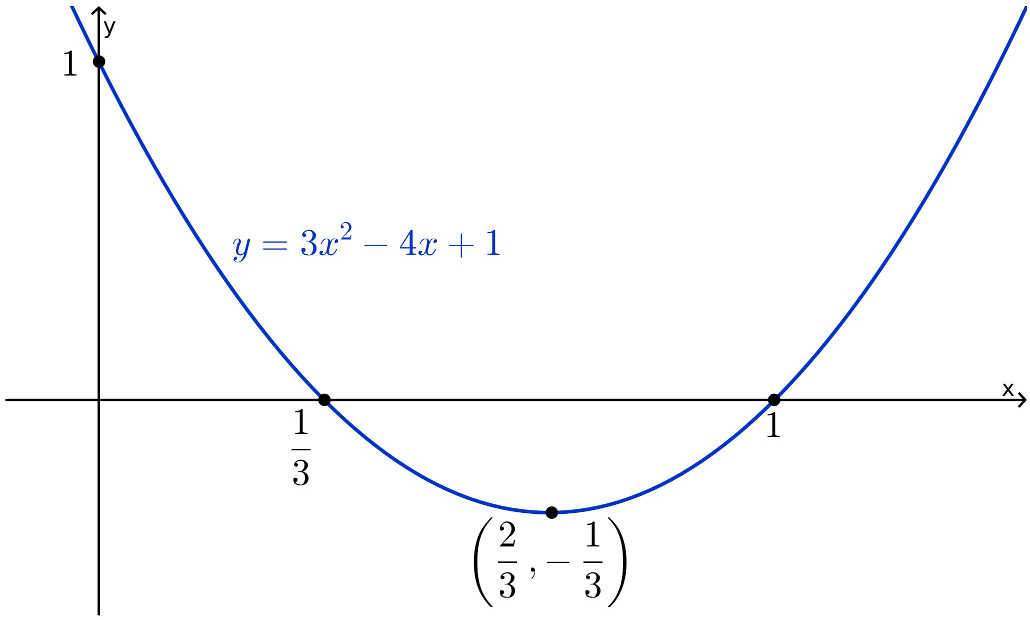


Figure 1: The Graph of the function $y = 3x^2 - 4x + 1$.

**4.** (a) (i) This is a function.
Its domain is $\mathbb{R}^+$ and its codomain is $\mathbb{R}^-$.

(ii) This is not a function.
For example, $f(-1)$ is not defined, since $\sqrt{-1} + 1 = i + 1$ does not lie in the codomain $\mathbb{R}$.

**[4]**

(b) Figure 2 shows the graph of the function

$$f\colon \{x \in \mathbb{R}\colon -2 \leqslant x \leqslant 2\} \to \{x \in \mathbb{R}\colon -10 \leqslant x \leqslant 10\}$$
$$x \mapsto 3x - 1$$

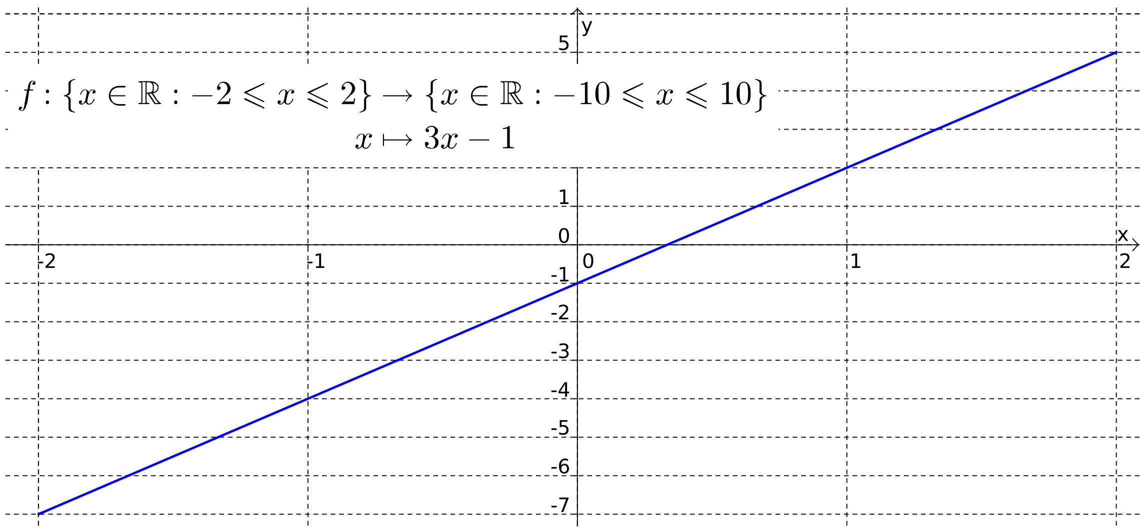


Figure 2: The graph of the function defined in Question 4(b).

**[2]**

(c) (i) This function is not injective since $f(2) = B = f(4)$.
It is not surjective since there is no $x$ with $f(x) = C$.
It is not bijective since it is neither injective nor surjective.

(ii) This function is injective, surjective and hence bijective.

**[3]**

(d) For the function in Part (c)(ii) we have $y = 2x - 3$, so

$$y = 2x - 3 \Rightarrow 2x = y + 3 \Rightarrow x = \frac{1}{2}y + \frac{3}{2}.$$

Hence its inverse function is

$$\begin{aligned} f \colon \mathbb{R} &\to \mathbb{R} \\ y &\mapsto \frac{1}{2}y + \frac{3}{2} \end{aligned}$$

which can also be written as

$$\begin{aligned} f \colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{1}{2}x + \frac{3}{2} \end{aligned}$$

**[2]**

(e)

$$\begin{aligned} 9^{2x} = 8 \quad &\iff \quad \ln\left(9^{2x}\right) = \ln(8) \quad \text{taking the natural log of each side} \\ &\iff \quad 2x\ln(9) = \ln(8) \quad \text{by Rule 2 of the Rules of Logs} \\ &\iff \quad x = \frac{\ln(8)}{2\ln(9)} \quad \text{dividing each side by } 2\ln(9). \end{aligned}$$

**[3]**

**5.** (a) $135^\circ = 135 \times \dfrac{\pi}{180} = \dfrac{3\pi}{4}$ Radians. **[1]**

(b) $\dfrac{5\pi}{3}$ Radians $= \left(\dfrac{5\pi}{3} \times \dfrac{180}{\pi}\right)^\circ = 300^\circ$. **[1]**

(c) In this case we want to find $\cos(\theta)$ when $\theta = \frac{7\pi}{6}$.

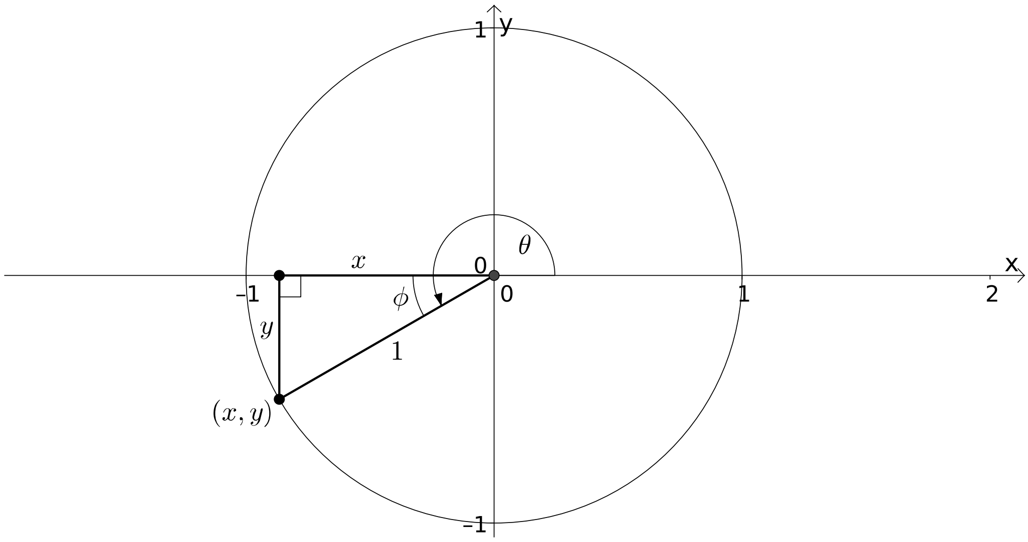


Figure 3: Calculation of $\cos\left(\frac{7\pi}{6}\right)$.

Looking at Figure 3, we see that we need to find $x$, since this is by definition $\cos\left(\frac{7\pi}{6}\right)$. Now, also from Figure 3, $\phi = \frac{7\pi}{6} - \pi = \frac{\pi}{6}$ (where we are just treating $\phi$ as an angle rather than a directed angle). Hence using the table of common values, $\cos(\phi) = \frac{\sqrt{3}}{2}$. But also by definition $\cos(\phi) = |x|$ (since the hypotenuse has length 1). Now, since $x$ is negative, $x = -|x|$ and so $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. **[3]**

(d) Using the sine rule in the form $\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$, we obtain $\frac{\sin(A)}{8} = \frac{\sin(44°)}{10}$. Thus $\sin(A) = \frac{8\sin(44°)}{10} \simeq 0.5557$. Hence $A \simeq 33.76°$ or $A \simeq 180° - 33.76° = 146.24°$. We now have to decide which of these values is correct. Suppose that $A \simeq 146.24°$. Then the two angles we know add up to approximately $44° + 146.24° = 190.24°$ and this is too many degrees for a triangle. Hence it must be that $A \simeq 33.76°$. **[3]**

(e) (i) We will use $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ with $A = \frac{\pi}{4}$ and $B = \frac{\pi}{3}$. Hence

$$\begin{aligned}\cos\left(\frac{7\pi}{12}\right) &= \cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)\\ &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{1-\sqrt{3}}{2\sqrt{2}}.\end{aligned}$$

(ii) Using $\tan^2(\theta) = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$ with $\theta = \dfrac{\pi}{8}$, we have

$$\tan^2\left(\frac{\pi}{8}\right) = \frac{1-\cos\left(2\times\frac{\pi}{8}\right)}{1+\cos\left(2\times\frac{\pi}{8}\right)} = \frac{1-\cos\left(\frac{\pi}{4}\right)}{1+\cos\left(\frac{\pi}{4}\right)} = \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}-1}{\sqrt{2}+1}.$$

Hence $\tan\left(\frac{\pi}{8}\right) = \sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}+1}}$.
We can also simplify this as follows:

$$\frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{2-2\sqrt{2}+1}{2-1} = 3-2\sqrt{2},$$

so that $\tan\left(\frac{\pi}{8}\right) = \sqrt{3-2\sqrt{2}}$.
In fact this can be further simplified to $\tan\left(\frac{\pi}{8}\right) = \sqrt{2}-1$
(it is easy to see $(\sqrt{2}-1)^2 = 3-2\sqrt{2}$ but not the other way around).
I will give full marks for $\tan\left(\frac{\pi}{8}\right) = \sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}+1}}$ however. **[4]**

**6.** (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{-2(x+h)+5-(-2x+5)}{h} \\ &= \lim_{h\to 0}\frac{-2x-2h+5+2x-5}{h} \\ &= \lim_{h\to 0}\frac{-2h}{h} \\ &= \lim_{h\to 0} -2 \\ &= -2. \end{aligned}$$

**[2]**

(b) (i) $f'(x) = 3x^{3-1} = 3x^2$.
(ii) $f'(x) = -4\cos(-4x)$.
(iii) $f'(x) = \dfrac{1}{3}\left(-\sin\left(\dfrac{1}{3}x\right)\right) = -\dfrac{1}{3}\sin\left(\dfrac{1}{3}x\right)$.
(iv) $f'(x) = -3\left(-\dfrac{1}{3}x^{-\frac{1}{3}-1}\right) + 2\left(-2e^{-2x}\right) - 5\left(\dfrac{1}{x}\right) = x^{-\frac{4}{3}} - 4e^{-2x} - \dfrac{5}{x}$. **[6]**

**7.** (a) $\displaystyle\int 2\,dx = 2x + c$. **[1]**

(b) $\displaystyle\int_{-1}^{1} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^{1} = \frac{1}{4}(1^4) - \frac{1}{4}(-1)^4 = 0.$ **[2]**

(c)

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \cos(3x)\ dx &= \left[\frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{6}}\\ &= \frac{1}{3}\sin\left(\frac{\pi}{2}\right) - \frac{1}{3}\sin(0)\\ &= \frac{1}{3}(1) - \frac{1}{3}(0)\\ &= \frac{1}{3}.\end{aligned}$$

**[2]**

(d)

$$\begin{aligned}\int 2e^{-4x} - 3x^{-1}\, dx &= 2\left(\frac{1}{-4}e^{-4x}\right) - 3\ln(x) + c\\ &= -\frac{1}{2}e^{-4x} - 3\ln(x) + c.\end{aligned}$$

**[2]**

**8.** (a) (i) The mean is

$$\begin{aligned}\overline{x} &= \frac{1}{8}(-3 + (-4) + (-9) + (-2) + (-5) + (-6) + (-6) + (-7))\\ &= \frac{-42}{8}\\ &= -\frac{21}{4}.\end{aligned}$$

(ii) The list in ascending order is $-9, -7, -6, -6, -5, -4, -3, -2$.
Since there are eight numbers (an even number), the median is
$m = \frac{x_{\frac{8}{2}} + x_{\frac{8}{2}+1}}{2} = \frac{x_4 + x_5}{2} = \frac{-6 + (-5)}{2} = -\frac{11}{2}$.

(iii) There are two minus sixes and one of each of the other numbers, the mode is $-6$.

(iv) Since there are eight numbers (an even number) we just split the numbers into a lower half $-9, -7, -6, -6$ and an upper half $-5, -4, -3, -2$. There are four numbers in each of these new groups (an even number), so the median of each group is $\frac{x_{\frac{4}{2}} + x_{\frac{4}{2}+1}}{2} = \frac{x_2 + x_3}{2}$.
Hence the lower quartile is $Q_1 = \frac{-7 + (-6)}{2} = -\frac{13}{2}$
and the upper quartile is $Q_3 = \frac{-4 + (-3)}{2} = -\frac{7}{2}$.
Thus the interquartile range is $Q_3 - Q_1 = -\frac{7}{2} - \left(-\frac{13}{2}\right) = 3$. **[5]**

(b) There are five points, so $n = 5$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{5} x_i = -4 + (-2) + 0 + 2 + 5 = 1$$
$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{5} y_i = 3 + 0 + 1 + (-2) + (-6) = -4$$

$$\begin{aligned}\sum_{i=1}^{n} x_i y_i =& \sum_{i=1}^{5} x_i y_i \\ =&(-4)(3) + (-2)(0) + (0)(1) + (2)(-2) + (5)(-6) \\ =& -12 + 0 + 0 - 4 - 30 \\ =& -46.\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n} x_i^2 =& \sum_{i=1}^{5} x_i^2 \\ =&(-4)^2 + (-2)^2 + 0^2 + 2^2 + 5^2 \\ =&16 + 4 + 0 + 4 + 25 \\ =&49.\end{aligned}$$

Hence

$$\begin{aligned} m &= \frac{n\left(\sum_{i=1}^{n} x_i y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2} \\ &= \frac{5(-46) - (1)(-4)}{5(49) - 1^2} \\ &= \frac{-226}{244} \\ &= -\frac{113}{122}, \end{aligned}$$

and

$$c = \overline{y} - m\overline{x} = \frac{\sum_{i=1}^{5} y_i}{5} - m\frac{\sum_{i=1}^{5} x_i}{5} = \frac{-4}{5} - \left(-\frac{113}{122}\right)\frac{1}{5} = -\frac{4}{5} + \frac{113}{610} = -\frac{75}{122}.$$

Thus the line of best fit is $y = -\frac{113}{122}x - \frac{75}{122}$. **[7]**